Formulas and Theorems for Reference

I. Trigonometric Formulas

1. $\sin^2\theta + \cos^2\theta = 1$

2. $1 + \tan^2\theta = \sec^2\theta$

3. $1 + \cot^2\theta = \csc^2\theta$

4. $\sin(-\theta) = -\sin\theta$

5. $\cos(-\theta) = \cos\theta$

6. $\tan(-\theta) = -\tan\theta$

7. $\sin(A+B) = \sin A\cos B + \sin B\cos A$

8. $\sin(A-B) = \sin A\cos B - \sin B\cos A$

9. $\cos(A+B) = \cos A\cos B - \sin A\sin B$

10. $\cos(A-B) = \cos A\cos B + \sin A\sin B$

11. $\sin 2\theta = 2\sin\theta\cos\theta$

12. $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$

13. $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{1}{\cot\theta}$

14. $\cot\theta = \dfrac{\cos\theta}{\sin\theta} = \dfrac{1}{\tan\theta}$

15. $\sec\theta = \dfrac{1}{\cos\theta}$

16. $\csc\theta = \dfrac{1}{\sin\theta}$

17. $\cos\left(\dfrac{\pi}{2} - \theta\right) = \sin\theta$

18. $\sin\left(\dfrac{\pi}{2} - \theta\right) = \cos\theta$

II. Differentiation Formulas

1. $\frac{d}{dx}(x^n) = nx^{n-1}$

2. $\frac{d}{dx}(fg) = fg' + gf'$

3. $\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{gf' - fg'}{g^2}$

4. $\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$

5. $\frac{d}{dx}(\sin x) = \cos x$

6. $\frac{d}{dx}(\cos x) = -\sin x$

7. $\frac{d}{dx}(\tan x) = \sec^2 x$

8. $\frac{d}{dx}(\cot x) = -\csc^2 x$

9. $\frac{d}{dx}(\sec x) = \sec x \tan x$

10. $\frac{d}{dx}(\csc x) = -\csc x \cot x$

11. $\frac{d}{dx}(e^x) = e^x$

12. $\frac{d}{dx}(a^x) = a^x \ln a$

13. $\frac{d}{dx}(\ln x) = \frac{1}{x}$

14. $\frac{d}{dx}(\text{ Arcsin } x) = \frac{1}{\sqrt{1 - x^2}}$

15. $\frac{d}{dx}(\text{ Arctan } x) = \frac{1}{1 + x^2}$

III. Integration Formulas

1. $\int a\, dx = ax + C$

2. $\int x^n\, dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$

3. $\int \frac{1}{x}\, dx = \ln|x| + C$

4. $\int e^x\, dx = e^x + C$

5. $\int a^x\, dx = \frac{a^x}{\ln a} + C$

6. $\int \ln x\, dx = x\ln x - x + C$

7. $\int \sin x\, dx = -\cos x + C$

8. $\int \cos x\, dx = \sin x + C$

9. $\int \tan x\, dx = \ln|\sec x| + C \text{ or } -\ln|\cos x| + C$

10. $\int \cot x\, dx = \ln|\sin x| + C$

11. $\int \sec x\, dx = \ln|\sec x + \tan x| + C$

12. $\int \csc x\, dx = \ln|\csc x - \cot x| + C$

13. $\int \sec^2 x\, dx = \tan x + C$

14. $\int \sec x \tan x\, dx = \sec x + C$

15. $\int \csc^2 x\, dx = -\cot x + C$

16. $\int \csc x \cot x\, dx = -\csc x + C$

17. $\int \tan^2 x\, dx = \tan x - x + C$

18. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\text{Arctan}\left(\frac{x}{a}\right) + C$

19. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \text{Arcsin}\left(\frac{x}{a}\right) + C$

IV. Formulas and Theorems

1. <u>Limits and Continuity</u>

A function $y = f(x)$ is continuous at $x = a$ if:

i) $f(a)$ is defined (exists)

ii) $\lim_{x \to a} f(x)$ exists, and

iii) $\lim_{x \to a} f(x) = f(a)$

Otherwise, f is discontinuous at $x = a$.

The limit $\lim_{x \to a} f(x)$ exists if and only if both corresponding one-sided limits exist and are equal — that is,

$$\lim_{x \to a} f(x) = L \Longleftrightarrow \lim_{x \to a^+} f(x) = L = \lim_{x \to a^-} f(x)$$

2. <u>Intermediate-Value Theorem</u>

A function $y = f(x)$ that is continuous on a closed interval $[a, b]$ takes on every value between $f(a)$ and $f(b)$.

<u>Note:</u> If f is continuous on $[a, b]$ and $f(a)$ and $f(b)$ differ in sign, then the equation $f(x) = 0$ has at least one solution in the open interval (a, b).

3. <u>Limits of Rational Functions as $x \to \pm\infty$</u>

1. $\lim_{x \to \pm\infty} \dfrac{f(x)}{g(x)} = 0$ if the degree of $f(x) <$ the degree of $g(x)$

<u>Example:</u> $\lim_{x \to \infty} \dfrac{x^2 - 2x}{x^3 + 3} = 0$

2. $\lim_{x \to \pm\infty} \dfrac{f(x)}{g(x)}$ is infinite if the degree of $f(x) >$ the degree of $g(x)$

<u>Example:</u> $\lim_{x \to +\infty} \dfrac{x^3 + 2x}{x^2 - 8} = \infty$

3. $\lim_{x \to \pm\infty} \dfrac{f(x)}{g(x)}$ is finite if the degree of $f(x) =$ the degree of $g(x)$

<u>Note:</u> The limit will be the ratio of the leading coefficient of $f(x)$ to $g(x)$.

<u>Example:</u> $\lim_{x \to \infty} \dfrac{2x^2 - 3x + 2}{10x - 5x^2} = -\dfrac{2}{5}$

4. Horizontal and Vertical Asymptotes

   1. A line $y = b$ is a horizontal asymptote of the graph of $y = f(x)$ if either $\lim_{x\to\infty} f(x) = b$ or $\lim_{x\to-\infty} f(x) = b$.

   2. A line $x = a$ is a vertical asymptote of the graph of $y = f(x)$ if either $\lim_{x\to a^+} f(x) = \pm\infty$ or $\lim_{x\to a^-} = \pm\infty$.

5. Average and Instantaneous Rate of Change

   1. Average Rate of Change: If (x_0, y_0) and (x_1, y_1) are points on the graph of $y = f(x)$, then the average rate of change of y with respect to x over the interval $[x_0, x_1]$ is

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y}{\Delta x}.$$

   2. Instantaneous Rate of Change: If (x_0, y_0) is a point on the graph of $y = f(x)$, then the instantaneous rate of change of y with respect to x at x_0 is $f'(x_0)$.

6. $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$ or $f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$

   The latter definition of the derivative is the instantaneous rate of change of $f(x)$ with respect to x at $x = a$.

   Geometrically, the derivative of a function at a point is the slope of the tangent line to the graph of the function at that point.

7. The Number e as a limit

   1. $\lim_{n\to+\infty} \left(1 + \frac{1}{n}\right)^n = e$

   2. $\lim_{n\to 0} (1+n)^{\frac{1}{n}} = e$

8. Rolle's Theorem

   If f is continuous on $[a, b]$ and differentiable on (a, b) such that $f(a) = f(b)$, then there is at least one number c in the open interval (a, b) such that $f'(c) = 0$.

9. Mean Value Theorem

   If f is continuous on $[a, b]$ and differentiable on (a, b), then there is at least one number c in (a, b) such that $\frac{f(b) - f(a)}{b - a} = f'(c)$.

10. Extreme-Value Theorem

    If f is continuous on a closed interval $[a, b]$, then $f(x)$ has both a maximim and a minimum on $[a, b]$.

11. To find the maximum and minimum values of a function $y = f(x)$, locate

    1. the points where $f'(x)$ is zero or where $f'(x)$ fails to exist
    2. the end points, if any, on the domain of $f(x)$.

    Note: These are the only candidates for the value of x where $f(x)$ may have a maximum or a minimum.

12. Let f be differentiable for $a < x < b$ and continuous for $a \leq x \leq b$.

    1. If $f'(x) > 0$ for every x in (a, b), then f is increasing on $[a, b]$.
    2. If $f'(x) < 0$ for every x in (a, b), then f is decreasing on $[a, b]$.

13. Suppose that $f''(x)$ exists on the interval (a, b).

    1. If $f''(x) > 0$ in (a, b), then f is concave upward in (a, b).
    2. If $f''(x) < 0$ in (a, b), then f is concave downward in (a, b).

    To locate the points of inflection of $y = f(x)$, find the points where $f''(x) = 0$ or where $f''(x)$ fails to exist. These are the only candidates where $f(x)$ may have a point of inflection. Then test these points to make sure that $f''(x) < 0$ on one side and $f''(x) > 0$ on the other.

14. If a function is differentiable at a point $x = a$, it is continuous at that point. The converse is false, i.e. continuity does not imply differentiability.

15. Linear Approximation

    The linear approximation to $f(x)$ near $x = x_0$ is given by $y = f(x_0) + f'(x_0)(x - x_0)$ for x sufficiently close to x_0.

    To estimate the slope of a graph at a point — just draw a tangent line to the graph at that point. Another way is (by using a graphics calculator) to "zoom in" around the point in question until the graph "looks" straight. This method almost always works. If we "zoom in" and the graph looks straight at a point, say $x = a$, then the function is locally linear at that point.

    The graph of $y = |x|$ has a sharp corner at $x = 0$. This corner cannot be smoothed out by "zooming in" repeatedly. Consequently, the derivative of $|x|$ does not exist at $x = 0$, hence, is not locally linear at $x = 0$.

16. Comparing Rates of Change

The exponential function $y = e^x$ grows very rapidly as $x \to \infty$ while the logarithmic function $y = \ln x$ grows very slowly as $x \to \infty$.

Exponential functions like $y = 2^x$ or $y = e^x$ grow more rapidly as $x \to \infty$ than polynomial or rational functions, and faster than any power of x, even $x^{100{,}000}$

Logarithmic functions like $y = \log_2 x$ or $y = \ln x$ grow more slowly as $x \to \infty$ than any positive power of x. The function $y = \ln x$ grows slower as $x \to \infty$ than any nonconstant polynomial.

We say, that as $x \to \infty$:

1. $f(x)$ grows faster than $g(x)$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$ or if $\lim_{x\to\infty} \frac{g(x)}{f(x)} = 0$.

   If $f(x)$ grows faster than $g(x)$ as $x \to \infty$, then $g(x)$ grows slower than $f(x)$ as $x \to \infty$.

2. $f(x)$ and $g(x)$ grow at the same rate as $x \to \infty$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L \neq 0$ (L is finite and nonzero).

For example,

1. e^x grows faster than x^3 as $x \to \infty$ since $\lim_{x\to\infty} \frac{e^x}{x^3} = \infty$
2. x^4 grows faster than $\ln x$ as $x \to \infty$ since $\lim_{x\to\infty} \frac{x^4}{\ln x} = \infty$
3. $x^2 + 2x$ grows at the same rate as x^2 as $x \to \infty$ since $\lim_{x\to\infty} \frac{x^2 + 2x}{x^2} = 1$

To find some of these limits as $x \to \infty$, you may use the graphing calculator. Make sure that an appropriate viewing window is used.

17. Inverse Functions

1. If f and g are two functions such that $f(g(x)) = x$ for every x in the domain of g, and, $g(f(x)) = x$, for every x in the domain of f, then, f and g are inverse functions of each other.
2. A function f has an inverse function if and only if no horizontal line intersects its graph more than once.
3. If f is either increasing or decreasing in an interval, then f has an inverse function over that interval.
4. If f is differentiable at every point on an interval I, and $f'(x) \neq 0$ on I, then $g = f^{-1}(x)$ is differentiable at every point of the interior of the interval $f(\text{I})$ and $g'(f(x)) = \frac{1}{f'(x)}$.

18. Properties of $y = e^x$

1. The exponential function $y = e^x$ is the inverse function of $y = \ln x$.
2. The domain is the set of all real numbers, $-\infty < x < \infty$.
3. The range is the set of all positive numbers, $y > 0$.
4. $\frac{d}{dx}(e^x) = e^x$.
5. $e^{x_1} \cdot e^{x_2} = e^{x_1 + x_2}$.
6. $y = e^x$ is continuous, increasing, and concave up for all x.
7. $\lim_{x \to +\infty} e^x = +\infty$ and $\lim_{x \to -\infty} e^x = 0$.
8. $e^{\ln x} = x$, for $x > 0$; $\ln(e^x) = x$ for all x.

19. Properties of $y = \ln x$

1. The domain of $y = \ln x$ is the set of all positive numbers, $x > 0$.
2. The range of $y = \ln x$ is the set of all real numbers, $-\infty < y < \infty$.
3. $y = \ln x$ is continuous and increasing everywhere on its domain.
4. $\ln(ab) = \ln a + \ln b$.
5. $\ln(a/b) = \ln a - \ln b$.
6. $\ln a^r = r \ln a$.
7. $y = \ln x < 0$ if $0 < x < 1$.
8. $\lim_{x \to +\infty} \ln x = +\infty$ and $\lim_{x \to 0^+} \ln x = -\infty$.
9. $\log_a x = \frac{\ln x}{\ln a}$.

20. Trapezoidal Rule

If a function f is continuous on the closed interval $[a, b]$ where $[a, b]$ has been partitioned into n subintervals $[x_0, x_1]$, $[x_1, x_2]$, $\ldots$, $[x_{n-1}, x_n]$, each of length $(b - a)/n$, then
$\int_a^b f(x)\,dx \approx \frac{b-a}{2n}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]$.

21. Properties of the Definite Integral

Let $f(x)$ and $g(x)$ be continuous on $[a,b]$.

1. $\int_a^b c \cdot f(x)\,dx = c\int_a^b f(x)\,dx$ for any constant c
2. $\int_a^a f(x)\,dx = 0$
3. $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$
4. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, where f is continuous on an interval containing the numbers a, b, and c
5. If $f(x)$ is an odd function, then $\int_{-a}^a f(x)\,dx = 0$
6. If $f(x)$ is an even function, then $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$
7. If $f(x) \geq 0$ on $[a,b]$, then $\int_a^b f(x)\,dx \geq 0$
8. If $g(x) \geq f(x)$ on $[a,b]$, then $\int_a^b g(x)\,dx \geq \int_a^b f(x)\,dx$

22a. Definition of Definite Integral as the Limit of a Sum

Suppose that a function $f(x)$ is continuous on the closed interval $[a,b]$. Divide the interval into n equal subintervals, of length $\Delta x = \dfrac{b-a}{n}$. Choose one number in each subinterval i.e. x_1 in the first, x_2 in the second, $\ldots$, x_k in the kth, $\ldots$, and x_n in the nth. Then $\lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x = \int_a^b f(x)\,dx = \mathrm{F}(b) - \mathrm{F}(a)$.

22b. Fundamental Theorem of Calculus:

$\int_a^b f(x)\,dx = F(b) - F(a)$, where $F'(x) = f(x)$, or $\dfrac{d}{dx}\int_a^x f(x)\,dx = f(x)$.

23. Velocity, Speed, and Acceleration

1. The velocity of an object tells how fast it is going and in which direction. Velocity is an instantaneous rate of change.
2. The speed of an object is the absolute value of the velocity, $|v(t)|$. It tells how fast it is going disregarding its direction.

   The speed of a particle increases (speeds up) when the velocity and acceleration have the same signs. The speed decreases (slows down) when the velocity and acceleration have opposite signs.
3. The acceleration is the instantaneous rate of change of velocity — it is the derivative of the velocity — that is, $a(t) = v'(t)$. Negative acceleration (deceleration) means that the velocity is decreasing. The acceleration gives the rate at which the velocity is changing.

Therefore, if x is the displacement of a moving object and t is time, then:

i) velocity $= v(t) = x'(t) = \dfrac{dx}{dt}$

ii) acceleration $= a(t) = x''(t) = v'(t) = \dfrac{dv}{dt} = \dfrac{d^2x}{dt^2}$

iii) $v(t) = \displaystyle\int a(t)\,dt$

iv) $x(t) = \displaystyle\int v(t)\,dt$

Note: The average velocity of a particle over the time interval from t_0 to another time t, is Average Velocity $= \dfrac{\text{Change in position}}{\text{Length of time}} = \dfrac{s(t) - s(t_0)}{t - t_0}$, where $s(t)$ is the position of the particle at time t.

24. The average value of $f(x)$ on $[a, b]$ is $\dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$.

25. If f and g are continuous functions such that $f(x) \geq g(x)$ on $[a, b]$, then the area between the curves is $\displaystyle\int_a^b [f(x) - g(x)]\,dx$.

26. Volumes of Solids of Revolution

Let f be nonnegative and continuous on $[a, b]$, and let R be the region bounded above by $y = f(x)$, below by the x-axis, and the sides by the lines $x = a$ and $x = b$.

1. When this region R is revolved about the x-axis, it generates a solid (having circular cross sections) whose volume $V = \int_a^b \pi[f(x)]^2 \, dx$.
2. When R is revolved about the y-axis, it generates a solid whose volume $V = \int_a^b 2\pi x f(x) \, dx$.

27. Volumes of Solids with Known Cross Sections

1. For cross sections of area $A(x)$, taken perpendicular to the x-axis, volume $= \int_a^b A(x) \, dx$.
2. For cross sections of area $A(y)$, taken perpendicular to the y-axis, volume $= \int_c^d A(y) \, dy$.